$$\alpha_{\text{opt}} = \frac{-B \pm [B^2 + 4H\,(JH/3P)^{1/2}]^{1/2}}{2H} \tag{11}$$

$$J = 2\pi R N \gamma_8 \qquad P = \frac{4\pi^{3/2}R^3 C_f^{1/2}\gamma_f}{k^{1/2}c^{3/4}NE_f^{1/2}}$$

Reference

[1] Shanley, F. R., *Weight-Strength Analysis of Aircraft Structures* (McGraw-Hill Book Co., Inc., New York, 1952), pp. 73–81.

Buckling of Shell-Supported Rings

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Nomenclature

A	= cross-sectional area of ring, in.^{2}
B	= point of support
C	= centroid
E	= modulus of elasticity, psi
G	= shear modulus, psi
I_x, I_y	= moment of inertia about x and y directions, respectively, in.^{4}
I_p	= polar moment inertia about z axis, in.^{4}
I_{xy}	= product of inertia, in.^{4}
J	= torsion constant (St. Venant), in.^{6}
M	= moment resultant, in.-lb/in.
N	= direct stress resultant, lb/in.
S	= shear center
U	= internal strain energy, in.-lb
V	= potential energy, in.-lb
W	= work done by external forces, in.-lb
k_x, k_y, k_z	= spring constant of support in x, y, and z directions, lb/in./in.
k_β	= rotational spring, lb/rad/in.
m	= integer
p	= external pressure, subscript for polar, psi
q	= external line of pressure, lb/in.
r, R	= radius, in.
u, v, w	= displacement component in x, y, z directions, respectively, in.
x, y, z	= coordinate directions and dimensions, in.
α	= half-apex angle of a conical support, rad
β	= rotation of cross section, rad
Γ	= warping constant
ϵ	= strain, in./in.
ϕ	= angular coordinate ($z = r_s\phi$), rad
θ	= twist, rad/in.
$\kappa_x, \kappa_y, \kappa_z$	= curvature change associated with moments about the x, y, and z coordinates, respectively, 1/in.
ρ	= $x + y \tan\alpha$, in.

Introduction

The purpose of this note is to develop ring formulas that will be useful in design and analysis of ring-stiffened shells. A ring stiffener is not a free ring for it is constrained by the attached shell. A first approximation of the shell constraint is to assume that the ring is rigidly supported in the direction of the shell meridian and has elastic support from the shell in the radial and tangential ring directions and in rotation.

The ring is loaded by an external line of pressure q which arises from the ring-shell discontinuity analysis.

The coordinates are x, measured inward from the shear center; y, normal to the plane of the ring; and $z = r_s\phi$ along the axis of shear centers. Corresponding displacements are u, v, and w, plus a rotation of the cross section β.

The constraint of the shell in the meridional direction requires that

$$v = (u - y_b\beta)\tan\alpha - x_b\beta \tag{1}$$

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where α is the half-apex angle of a cone tangent to the middle surface of the shell at the point of attachment of the ring. Figure 1 indicates the geometry of this support. It is always possible to represent the elastic support by a set of springs k_x, k_β, and k_z attached at x_b, $y_b = y_c$. The external line of pressure q is applied at this same point in order that the prebuckling stress distribution be free of primary bending stress. The total potential energy expression with the constraint of Eq. (1) becomes[1]

$$V = \tfrac{1}{2}\int \Bigg\{EA\epsilon_0^2 + EI_x\kappa_x^2 + EI_y\kappa_y^2 - 2EI_{xy}\kappa_x\kappa_y + GJ\theta^2 + \frac{E\Gamma}{r_s^2}\theta'^2 + \Bigg[k_x(u - y_c\beta)^2 + k_\beta\beta^2 + k_z \times \left(w\frac{r_b}{r_s} - \frac{x_b u'}{r_s} - \frac{y_c}{r_s}(u'\tan\alpha - \rho_b\beta')\right)^2\Bigg]\frac{r_b}{r_c} - \frac{q}{r_c} \times \Bigg[u'^2 + (u'\tan\alpha - \rho_b\beta')^2 - u^2 - 2y_c(u'\beta' - u\beta) + 2x_c \times (u'\tan\alpha - \rho_b\beta')\beta' + \frac{I_p}{A}\beta'^2 - \frac{(I_x + y_c^2A)}{A}\beta^2 - r_c \times (x_c - x_b\beta^2\Bigg]\Bigg\} r_c d\phi \tag{2}$$

where

$$\epsilon_0 = \frac{w'}{r_s} - \frac{\rho_c}{r_c r_s}u'' - \frac{u}{r_c} + \frac{y_c\rho_b}{r_c r_s}\beta'' + \frac{y_c}{r_c}\beta$$

$$\kappa_x = \frac{\beta}{r_c} - \frac{(u''\tan\alpha - \rho_b\beta'')}{r_c r_s}$$

$$\kappa_y = \frac{u''}{r_c^2} + \frac{y_c}{r_c^2 r}(u''\tan\alpha - \rho_b\beta'') + \frac{u}{r_c^2} - \frac{y_c}{r_c^2}\beta$$

$$\theta = \frac{\beta'}{r_s} + \frac{(u'\tan\alpha - \rho_b\beta')}{r_s^2}$$

$$\rho_b = x_b + y_c\tan\alpha$$

$$\rho_c = x_c + y_c\tan\alpha$$

The procedure is to set the first variation of the potential energy equal to zero. The fundamental lemma of the calculus of variations leads to the equations of equilibrium. A solution to the equations of equilibrium of the form,

$$u = u_0 \sin m\phi \tag{3}$$

$$w = w_0 \cos m\phi \tag{4}$$

$$\beta = \beta_0 \sin m\phi \tag{5}$$

may be assumed, which satisfies the requirement in a ring that the displacements be periodic on ϕ. Upon substitution of

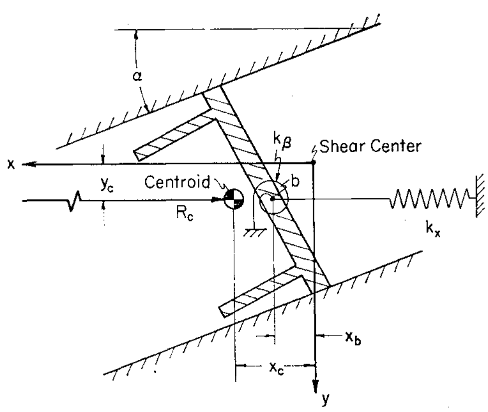

Fig. 1 Coordinates used and constraints approximating the supporting shell.

Table 1 Critical buckling pressure for condition $I_p/Ar_c = x_c = y_c = x_b = 0$

Condition	Critical pressure
1. General	$q = \dfrac{1}{r^3(m^2\sec^2\alpha - 1)}\left[EI_y(m^2-1)^2 + EI_x m^4\tan^2\alpha + 2EI_{xy}m^2(m^2-1)\tan\alpha + GJ\,m^2\tan^2\alpha + \dfrac{E\Gamma}{r^2}m^4\tan^2\alpha + k_x r^4 + \dfrac{k_z r^4}{\left(m^2 + \dfrac{k_z r^2}{EA}\right)} - \dfrac{(EI_x\,m^2\tan\alpha + EI_{xy}(m^2-1) + GJ\,m^2\tan\alpha + (E\Gamma/r^2)\,m^4\tan\alpha)^2}{EI_x + GJ\,m^2 + (E\Gamma/r^2)\,m^4 + k_\beta}\right]$
2. $EI_{xy} = k_\beta = 0$	$q = \dfrac{1}{r^3(m^2\sec^2\alpha - 1)}\left[EI_y(m^2-1)^2 + \dfrac{EI_x m^2\tan^2\alpha(m^2-1)^2\left(GJ + \dfrac{E\Gamma m^2}{r^2}\right)}{(E\Gamma/r^2)\,m^4 + GJm^2 + EI_x} + k_x r^4 + \dfrac{k_z r^4}{\left(m^2 + \dfrac{k_z r^2}{EA}\right)}\right]$
3. $GJ = E\Gamma = k_\beta = 0$	$q = \dfrac{1}{r^3(m^2\sec^2\alpha - 1)}\left[EI_y(m^2-1)^2\left(1 - \dfrac{I_{xy}^2}{I_x I_y}\right) + k_x r^4 + \dfrac{k_z r^4}{\left(m^2 + \dfrac{k_z r^2}{EA}\right)}\right]$
4. $k_\beta = k_x = k_z = 0$ $EI_{xy} = 0$ $m_{\text{crit}} = 2$	$q = \dfrac{1}{r^3(4\sec^2\alpha - 1)}\left[9EI_y + \dfrac{36EI_x\tan^2\alpha\left(GJ + \dfrac{4E\Gamma}{r^2}\right)}{16\,E\Gamma/r^2 + 4GJ + EI_x}\right]$

Eq. (3–5), the three equations of equilibrium may be placed in the following matrix form:

$$\begin{vmatrix} C_1 + C_2 q & C_3 & C_4 + C_5 q \\ C_3 & C_6 & C_7 \\ C_4 + C_5 q & C_7 & C_8 + C_9 q \end{vmatrix} \begin{Bmatrix} u \\ w \\ \beta \end{Bmatrix} = 0 \quad (6)$$

where

$$C_1 = \frac{EA}{r_c^2}B_1^2 + \frac{EI_x\tan^2\alpha}{r_c^2 r_s^2}m^4 + \frac{EI_y}{r_c^4}B_2^2 + \frac{2EI_{xy}\tan\alpha}{r_c^3 r_s}m^2B_2 + \frac{GJ\tan^2\alpha}{r_s^4}m^2 + \frac{E\Gamma\tan^2\alpha}{r_s^6}m^4 + \frac{k_x r_b}{r_c} + \frac{k_z\rho_b^2 r_b}{r_s^2 r_c}m^2 \quad (7)$$

$$C_2 = -(m^2\sec^2\alpha - 1)/r_c \quad (8)$$

$$C_3 = -\frac{EA}{r_c r_s}mB_1 - \frac{k_z\rho_b r_b^2}{r_s^2 r_c}m \quad (9)$$

$$C_4 = -\frac{EAy_c}{r_c}B_1\cdot B_3 - \frac{EI_x\tan\alpha}{r_c^2 r_s}m^2B_3 - \frac{EI_y y_c}{r_c^4}B_2\cdot B_3 - \frac{EI_{xy}}{r_c^3}B_3\cdot B_4 + \frac{GJ\tan\alpha}{r_s^3}m^2B_5 + \frac{E\Gamma\tan\alpha}{r_s^5}m^4B_5 - \frac{k_x r_b y_c}{r_c} - \frac{k_z y_c\rho_b^2 r_b}{r_s^2 r_c}m^2 \quad (10)$$

$$C_5 = \frac{y_c(m^2-1)}{r_c} - \frac{(\rho_b - x_c)\tan\alpha}{r_c}m^2 \quad (11)$$

$$C_6 = \frac{EA}{r_s^2}m^2 + \frac{k_z r_b^3}{r_s^2 r_c} \quad (12)$$

$$C_7 = \frac{EAy_c}{r_s r_c}mB_3 + \frac{k_z y_c\rho_b r_b^2}{r_s^2 r_c}m \quad (13)$$

$$C_8 = \left(\frac{EAy_c^2}{r_c^2} + \frac{EI_x}{r_c^2} + \frac{EI_y y_c^2}{r_c^4} + \frac{2EI_{xy}y_c}{r_c^3}\right)B_3^2 + \left(\frac{GJm^2}{r_s^2} + \frac{E\Gamma m^4}{r_s^4}\right)B_5^2 + \frac{k_x y_c^2 r_b}{r_c} + \frac{k_\beta r_b}{r_c} + \frac{k_z y_c^2\rho_b r_b}{r_c r_s^2}m^2 \quad (14)$$

$$C_9 = -\left(\frac{\rho_b^2}{r_c} + \frac{I_p}{Ar_c} - \frac{2x_c\rho_b}{r_c}\right)m^2 + \frac{I_x + y_c 2A}{Ar_c} + x_c - x_b \quad (15)$$

and

$$B_1 = (\rho_c/r_s)m^2 - 1 \quad (16)$$

$$B_2 = \frac{r_s + y_c\tan\alpha}{r_s}m^2 - 1 \quad (17)$$

$$B_3 = (\rho_b/r_s)m^2 - 1 \quad (18)$$

$$B_4 = \frac{r_s + 2y_c\tan\alpha}{r_s}m^2 - 1 \quad (19)$$

$$B_5 = 1 - (\rho_b/r_s) \quad (20)$$

Buckling Pressure

The determinant of the C matrix must be zero if the displacements are not. The determinant in this case is a quadratic in q. The solution is the smaller value of

$$q = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a} \quad (21)$$

where

$$a = C_6(C_2\cdot C_9 - C_5^2)$$
$$b = C_6(C_1\cdot C_9 + C_2\cdot C_8 - 2C_4\cdot C_5) + 2C_3C_5C_7 - C_2C_7 - C_9C_3^2$$
$$c = C_6(C_1\cdot C_8 - C_4^2) + 2C_3\cdot C_7C_4 - C_1\cdot C_7^2 - C_1\cdot C_7^2 - C_3^2\cdot C_8$$

The numerical determination of the critical buckling pressure involves a trial and error solution of Eq. (21) to ascertain the value of m that yields the smallest value of q.

The solution becomes simplified if some of the parameters are zero. If, for example, the tangential spring support, represented in k_z, is zero, the axial stiffness EA may be eliminated for the expressions. The coefficients C_3, C_6, and C_7 become

$$C_3 = -\frac{1}{r_c}\left(\frac{\rho_c}{r_s}m^2 - 1\right) \quad (22)$$

$$C_6 = m/r_s \quad (23)$$

$$C_7 = \frac{y_c}{r_s}\left(\frac{\rho_b}{r_s}m^2 - 1\right) \quad (24)$$

and the EA terms in the other coefficients are dropped. This shows that the usual assumption of inextensional buckling is correct if there are no tangential loads or supports.

Greater simplification may be made if the coefficient a in Eq. (21) is zero, for then the quadratic degenerates and the modes of buckling become uncoupled. This occurs if $x_c = y_c = x_b = I_x/Ar_c = I_p/Ar_c = 0$. Since $I_p = 0$ requires that both I_x and I_x be zero, the solution would not be very significant from a design standpoint. However, if $Ar_c \gg I_p$, approximate solutions may be obtained by neglecting the terms I_x/Ar_c and I_x/Ar_c. Therefore, with $a = 0$,

$$q = \frac{1}{C_2}\left(-C_1 + \frac{C_3^2}{C_7} + \frac{C_4^2}{C_{12}}\right) \tag{25}$$

The determination of the effective elastic support for the ring is the subject of future research. However, in general the spring support may be obtained by choosing a spring constant that produces the same internal strain energy as the shell does upon distortion.

A formulation of an effective elastic spring constant is made by Czerwenka[2] for a shell beyond initial buckling. The value is in general a function of the buckling mode m.

Conclusion

The buckling pressure for a shell-supported ring has been derived and is represented by Eq. (21). The equation may be simplified in the case where the shear center S coincides with the centroid C, the loading is through the centroid, and $Ar_c \gg I_p$. The result of such simplification is given in Eq. (26). Table 1 gives explicit equations derived from the latter case.

References

[1] Cheney, J. A., "Bending and buckling of rings of thin-walled open section," Proc. Am. Soc. Civil Engrs: J. Eng. Mech. Div. **89**, 17–14 (October 1963).

[2] Czerwenka, G., "Untersuchugen von dünnen kurzen Zylindern, die durch Ring-Kleinst profile enger and mittlerer Teilung verstärkt sind und unter Manteldruck stehen," Z. Flugwiss. **9**, 163–190 (1961).

Technical Comments

Comment on "Low-Altitude, High-Speed Handling and Riding Qualities"

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The following offers an explanation for the "PIO limits" reported by A'Harrah[1] and uncovers a common deficiency in ground and flight simulation studies of handling qualities, one which can easily be avoided when appreciated.

For a sustained Pilot-Induced Oscillation (PIO) it is postulated that, because of the regular nature of the oscillation seen or felt by the pilot, he will synchronize his output and eliminate his usual reaction-time delay.[2] Under such circumstances the pilot's sinusoidal describing function is a simple gain. Assuming that his primary response is to visual pitch-attitude cues, the pertinent longitudinal pilot-vehicle open-loop describing function (neglecting phugoid motions) at any limit cycling frequency ω, where $s = j\omega$, is [3]

$$Y_p(s)\frac{\theta}{\delta_e}(s) = \frac{K_pM_\delta[s + (1/T_{\theta_2})]}{s(s^2 + 2\zeta_{sp}\omega_{sp}s + \omega_{sp}^2)} \tag{1}$$

Accordingly, the closed loop characteristic equation is given by

$$\begin{aligned}\Delta' &= s(s^2 + 2\zeta_{sp}\omega_{sp}s + \omega_{sp}^2) + K_pM_\delta[s + (1/T_{\theta_2})]\\ &= s^3 + s^2(2\zeta_{sp}\omega_{sp}) + s(\omega_{sp}^2 + K_pM_\delta) + K_pM_\delta(1/T_{\theta_2})\end{aligned} \tag{2}$$

where the usual factored form of Δ' is expressed as

$$\begin{aligned}\Delta' &= [s + (1/T_c)](s^2 + 2\zeta_{sp}'\omega_{sp}'s + \omega_{sp}'^2)\\ &= s^3 + s^2[(1/T_c) + 2\zeta_{sp}'\omega_{sp}'] + s[\omega_{sp}'^2 + (2\zeta_{sp}'\omega_{sp}'/T_c)] + \omega_{sp}'^2/T_c\end{aligned} \tag{3}$$

Since a sustained oscillation implies $\zeta' = 0$, the conditions for PIO [and the only condition for which Eq. (3) has meaning] correspond to those for which $\zeta_{sp}'\omega_{sp}'$ is zero. That is equating the s^2 coefficients of Eqs. (2) and (3),

$$\begin{aligned}2\zeta_{sp}'\omega_{sp}' + (1/T_c) &= 2\zeta_{sp}\omega_{sp}\\ 2\zeta_{sp}'\omega_{sp}' &= 2\zeta_{sp}\omega_{sp} - (1/T_c) = 0\end{aligned} \tag{4}$$

Obviously if $1/T_c$ can become equal to $2\zeta_{sp}\omega_{sp}$, the system *can* be driven unstable by sufficiently high gain. To get a better feeling for such possibilities, consider the root locus plot of Eq. (1) given in Fig. 1. The relationship of Eq. (4) is graphically illustrated here, and conclusions as to the maximum value of $1/T_c$ and minimum value of $2\zeta_{sp}'\omega_{sp}'$ are clearly

$$\begin{aligned}(1/T_c)_{\max} &\to (1/T_{\theta_2})\\ (2\zeta_{sp}'\omega_{sp}')_{\min} &\to 2\zeta_{sp}\omega_{sp} - (1/T_{\theta_2})\end{aligned} \tag{5}$$

Probable values for the right side of Eq. (5) can be obtained by considering the approximate factors of Ref. 4,

$$\begin{aligned}2\zeta_{sp}\omega_{sp} &\doteq -(Z_w + M_q + M_{\dot\alpha})\\ (1/T_{\theta_2}) &\doteq -Z_w + (Z_\delta/M_\delta)M_w\end{aligned} \tag{6}$$

so that

$$\begin{aligned}2\zeta_{sp}\omega_{sp} - (1/T_{\theta_2}) &\doteq -M_q - M_{\dot\alpha} - (Z_\delta/M_\delta)M_w\\ &\doteq \frac{\rho S U_0 c^2}{4I_y}\left[-C_{M_q} - C_{M_{\dot\alpha}} + \frac{2k_y^2}{cl_\delta}C_{M_\alpha}\right]\end{aligned} \tag{7}$$

where $l_\delta = cC_{M_\delta}/C_{L_\delta}$ is the effective elevator control arm measured positive *forward*. For conventional tail-aft airplanes with some static margin (l_δ and C_{M_α} both negative) the bracketed terms of Eq. (7) are always positive; for canard control the contribution of the C_{M_α} term will be negative because of the positive l_δ. In the latter case it is conceivable that the entire right side of Eq. (7) could be negative. However, a general observation is that it will take a very unusual configuration with a small l_δ and low values of $-C_{M_q}$ (which, for conventional airplanes, is usually an order of magnitude greater than C_{M_α}), etc., to make the value of $2\zeta_{sp}\omega_{sp} - 1/T_{\theta_2}$ negative, as sketched in Fig. 1. Therefore, *only* for such unusual configurations is there a possibility of driving $\zeta'\omega'$ to zero to achieve a sustained PIO.

The foregoing demonstrates that for real airplanes with negligible control system dynamics (including nonlinear ele-

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